

# Pseudo- $\epsilon$ Expansion and Renormalized Coupling Constants at Criticality

A. I. Sokolov\* and M. A. Nikitina

*Department of Quantum Mechanics,*

*Saint Petersburg State University, Ulyanovskaya 1,*

*Petergof, Saint Petersburg, 198504 Russia*

(Dated: February 19, 2014)

## Abstract

Universal values of dimensional effective coupling constants  $g_{2k}$  that determine nonlinear susceptibilities  $\chi_{2k}$  and enter the scaling equation of state are calculated for  $n$ -vector field theory within the pseudo- $\epsilon$  expansion approach. Pseudo- $\epsilon$  expansions for  $g_6$  and  $g_8$  at criticality are derived for arbitrary  $n$ . Analogous series for ratios  $R_6 = g_6/g_4^2$  and  $R_8 = g_8/g_4^3$  figuring in the equation of state are also found and the pseudo- $\epsilon$  expansion for Wilson fixed point location  $g_4^*$  descending from the six-loop RG expansion for  $\beta$ -function is reported. Numerical results are presented for  $0 \leq n \leq 64$  with main attention paid to physically important cases  $n = 0, 1, 2, 3$ . Pseudo- $\epsilon$  expansions for quartic and sextic couplings have rapidly diminishing coefficients, so Padé resummation turns out to be sufficient to yield high-precision numerical estimates. Moreover, direct summation of these series with optimal truncation gives the values of  $g_4^*$  and  $R_6^*$  almost as accurate as those provided by Padé technique. Pseudo- $\epsilon$  expansion estimates for  $g_8^*$  and  $R_8^*$  are found to be much worse than that for the lower-order couplings independently on the resummation method employed.

Numerical effectiveness of the pseudo- $\epsilon$  expansion approach in two dimensions is also studied. Pseudo- $\epsilon$  expansion for  $g_4^*$  originating from the five-loop RG series for  $\beta$ -function of 2D  $\lambda\phi^4$  field theory is used to get numerical estimates for  $n$  ranging from 0 to 64. The approach discussed gives accurate enough values of  $g_4^*$  down to  $n = 2$  and leads to fair estimates for Ising and polymer ( $n = 0$ ) models.

PACS numbers: 05.10.Cc, 05.70.Jk, 64.60.ae, 64.60.Fr

## I. INTRODUCTION

The behavior of physical systems in the vicinity of Curie point is characterized by universal parameters such as critical exponents, critical amplitude ratios, etc. Among universal quantities important role is played by renormalized effective coupling constants  $g_{2k}$  which enter small magnetization expansion of free energy and fix critical asymptotes of nonlinear susceptibilities:

$$F(z, m) - F(0, m) = \frac{m^3}{g_4} \left( \frac{z^2}{2} + z^4 + \frac{g_6}{g_4^2} z^6 + \frac{g_8}{g_4^3} z^8 + \dots \right), \quad (1)$$

$$\begin{aligned} \chi_4 = \left. \frac{\partial^3 M}{\partial H^3} \right|_{H=0} &= -24 \frac{\chi^2}{m^3} g_4, & \chi_6 = \left. \frac{\partial^5 M}{\partial H^5} \right|_{H=0} &= 720 \frac{\chi^3}{m^6} (8g_4^2 - g_6), \\ \chi_8 = \left. \frac{\partial^7 M}{\partial H^7} \right|_{H=0} &= -40320 \frac{\chi^4}{m^9} (96g_4^3 - 24g_4g_6 + g_8), \end{aligned} \quad (2)$$

where  $z$  is dimensionless magnetization,  $z = M \sqrt{g_4/m^{1+\eta}}$ , renormalized mass  $m \sim (T - T_c)^\nu$  being the inverse correlation length,  $\chi$  is a linear susceptibility,  $\chi_4$ ,  $\chi_6$  and  $\chi_8$  – nonlinear susceptibilities of fourth, sixth and eighth orders.

Nonlinear susceptibilities and critical equation of state, describing the influence of ordering field upon the behavior of a system near  $T_c$ , attract permanent attention of theorists for decades. Dimensionless effective coupling constants  $g_{2k}$  and free energy (effective action) for the basic models of phase transitions were found by a number of analytical and numerical methods<sup>1–38</sup>. Calculation of the universal critical values of  $g_4$  and  $g_6$  for three-dimensional Ising model showed that the field-theoretical renormalization group (RG) approach in fixed dimensions yields highly accurate numerical estimates for these quantities. For example, the resummation of four- and five-loop RG expansions by means of the Borel-transformation-based procedures gave the values for  $g_6^*$ , which differ from each other by less than 0.5%<sup>15,16</sup> while use of the resummed three-loop RG expansion enabled one to achieve an apparent accuracy no worse than 1.6%<sup>15,24</sup>. In principle, this is not surprising since the field-theoretical RG approach proved to be highly efficient when used to estimate critical exponents, critical amplitude ratios, marginal dimensionality of the order parameter, etc. for various three-dimensional models<sup>2,3,10,19,27,32,33,39,40</sup>. Moreover, the field-theoretical RG technique turns out to be powerful enough even in two dimensions: properly resummed four-loop<sup>2,3</sup> and

five-loop<sup>41</sup> RG expansions lead to fair numerical estimates for the critical exponents and renormalized coupling constant  $g_6^{*23}$  of 2D Ising model and give reasonable results for other exactly solvable 2D models<sup>42–44</sup>.

Since RG expansions are known to be divergent, in order to obtain proper numerical estimates one needs in resummation methods. Most of those nowadays being employed are based on Borel transformation which avoids factorial growth of higher-order coefficients and paves the way to converging iteration schemes. This transformation widely used in the theory of critical phenomena has resulted in a great number of high precision numerical estimates. On the other hand, there exists alternative technique turning divergent series into more "friendly" ones in the sense that the expansions this technique yields have smaller lower-order coefficients and much slower growing higher-order ones than those of original series. We mean the method of pseudo- $\epsilon$  expansion put forward by B. Nickel many years ago (see Ref. 19 in the paper of Le Guillou and Zinn-Justin<sup>3</sup>).

Pseudo- $\epsilon$  expansion approach was shown to be rather effective when used to estimate numerical values of critical exponents and some other universal quantities characterizing critical behavior of three-dimensional systems<sup>3,19,45–47</sup>. In two dimensions, where original RG series are shorter and more strongly divergent, pseudo- $\epsilon$  expansion technique is also able to give good or satisfactory results<sup>3,36,42,44</sup>. To obtain numerical estimates from pseudo- $\epsilon$  expansions one has to apply a resummation technique since corresponding series have growing higher-order coefficients, i. e. remain divergent. However, in contrast to RG expansions in fixed and  $4 - \epsilon$  dimensions, pseudo- $\epsilon$  expansions do not require advanced resummation procedures based on Borel transformation. As a rule, use of simple Padé approximants<sup>36,42,44,45</sup> or even direct summation with an optimal cut off<sup>44</sup> turn out to be sufficient to obtain fair numerical estimates.

In this paper, we study renormalized effective coupling constants and corresponding universal ratios of three-dimensional  $O(n)$ -symmetric systems within the frame of pseudo- $\epsilon$  expansion technique. The pseudo- $\epsilon$  expansions ( $\tau$ -series) for renormalized coupling constants  $g_6$  and  $g_8$  will be calculated on the base of four-loop and three-loop RG expansions obtained earlier<sup>24,25</sup> for 3D  $n$ -vector field theory of  $\lambda\varphi^4$  type. Along with the higher-order couplings, Wilson fixed point coordinate  $g_4^*$  and universal critical values of ratios  $R_6 = g_6/g_4^2$ ,  $R_8 = g_8/g_4^3$  will be found as series in  $\tau$  up to  $\tau^6$ ,  $\tau^4$  and  $\tau^3$  terms respectively. The pseudo- $\epsilon$  expansions obtained will be analyzed for  $n = 0$ ,  $n = 1$ ,  $n = 2$ , and  $n = 3$ , i. e. for the

systems most interesting from the physical point of view, as well as for other values of  $n$  ranging up to 64. In this context, Ising, XY and Heisenberg models, apart from their physical importance, may be considered as testbeds for clarification of numerical accuracy of the pseudo- $\epsilon$  expansion machinery since today we have a big amount of alternative estimates for renormalized effective couplings obtained within the  $\epsilon$ -expansion technique, on the base of perturbative RG expansions in physical dimensions and extracted from lattice calculations and computer simulations. Renormalized quartic coupling for the two-dimensional  $n$ -vector model will be also studied for various  $n$  and comparison of the results given by pseudo- $\epsilon$  expansion approach with their field-theoretical and lattice counterparts will be made. Numerical estimates will be deduced from the pseudo- $\epsilon$  expansions by means of Padé and, when necessary, Padé-Borel resummation methods as well as by direct summation. The latter approach will be applied under the assumption that proper numerical results may be obtained by means of truncating divergent pseudo- $\epsilon$  expansions at smallest terms, i. e. applying the procedure true for asymptotic series.

The paper is organized as follows. In the next section the pseudo- $\epsilon$  expansions for  $g_4^*$ ,  $g_6^*$ ,  $g_8^*$ ,  $R_6^*$ , and  $R_8^*$  are derived from 3D RG series for general  $n$ . Section III contains numerical estimates of Wilson fixed point location for 3D  $O(n)$ -symmetric systems with  $0 \leq n \leq 64$  resulting from pseudo- $\epsilon$  expansion for  $g_4^*$ . Sections IV and V deal with renormalized sextic and octic couplings respectively presenting and discussing numerical estimates for  $R_6^*$  and  $R_8^*$  at various  $n$ . In Section VI the quartic coupling constant of two-dimensional  $n$ -vector model is studied within the pseudo- $\epsilon$  expansion approach. The last section is a summary of the results obtained.

## II. PSEUDO- $\epsilon$ EXPANSIONS FOR QUARTIC, SEXTIC AND OCTIC COUPLING CONSTANTS

As is well known, the critical behavior of  $D$ -dimensional systems with  $O(n)$ -symmetric vector order parameters may be described by Euclidean field theory with the Hamiltonian:

$$H = \int d^D x \left[ \frac{1}{2} (m_0^2 \varphi_\alpha^2 + (\nabla \varphi_\alpha)^2) + \frac{\lambda}{24} (\varphi_\alpha^2)^2 \right], \quad (3)$$

where  $\varphi_\alpha$  is a real  $n$ -vector field, bare mass squared  $m_0^2$  being proportional to  $T - T_c^{(0)}$ ,  $T_c^{(0)}$  – mean field transition temperature. The  $\beta$ -function for the model (3) in three dimensions

have been calculated within the massive theory<sup>2,48</sup> with the propagator, quartic vertex and  $\varphi^2$  insertion normalized in a standart way:

$$\begin{aligned} G_R^{-1}(0, m, g_4) &= m^2, & \frac{\partial G_R^{-1}(p, m, g_4)}{\partial p^2} \Big|_{p^2=0} &= 1, \\ \Gamma_R(0, 0, 0, m, g) &= m^2 g_4, & \Gamma_R^{1,2}(0, 0, m, g_4) &= 1. \end{aligned} \quad (4)$$

Starting from the six-loop 3D RG expansion for  $\beta$ -function<sup>10</sup>, we replace the linear term in this expansion with  $\tau g_4$ , calculate the Wilson fixed point coordinate as series in  $\tau$ , and arrive to the following expression:

$$\begin{aligned} g_4^* &= \frac{2\pi}{n+8} \left[ \tau + \frac{\tau^2}{(n+8)^2} \left( 6.074074074 n + 28.14814815 \right) \right. \\ &+ \frac{\tau^3}{(n+8)^4} \left( -1.34894276 n^3 + 8.056832799 n^2 + 44.73231547 n - 12.48684745 \right) \\ &+ \frac{\tau^4}{(n+8)^6} \left( -0.15564589 n^5 - 7.638021730 n^4 + 100.0250844 n^3 + 679.8756744 n^2 \right. \\ &+ 1604.099837 n + 3992.366079 \left. \right) \\ &- \frac{\tau^5}{(n+8)^8} \left( 0.05123618 n^7 + 4.68103281 n^6 + 80.8238429 n^5 - 176.369063 n^4 \right. \\ &+ 11347.4861 n^3 + 153560.921 n^2 + 646965.181 n + 963077.072 \left. \right) \left. \right] \\ &+ \frac{\tau^6}{(n+8)^{10}} \left( -0.0234242 n^9 - 2.5301565 n^8 - 71.923926 n^7 + 1183.9160 n^6 + 59058.036 n^5 \right. \\ &+ 631059.29 n^4 + 3909462.7 n^3 + 17512239 n^2 + 50941121 n + 66886678 \left. \right) \left. \right]. \quad (5) \end{aligned}$$

Substituting this expansion into four-loop RG series for sextic coupling constant  $g_6$ <sup>24,25</sup>

$$\begin{aligned} g_6 &= \frac{9}{\pi} g_4^3 \left[ \frac{n+26}{27} - \frac{17 n + 226}{81\pi} g_4 + (0.000999164 n^2 + 0.14768927 n + 1.24127452) g_4^2 \right. \\ &- \left. (-0.00000949 n^3 + 0.00783129 n^2 + 0.34565683 n + 2.14825455) g_4^3 \right] \quad (6) \end{aligned}$$

and into three-loop series for  $g_8$ <sup>25</sup>

$$\begin{aligned} g_8 &= -\frac{81}{2\pi} g_4^4 \left[ \frac{n+80}{81} - \frac{81 n^2 + 7114 n + 134960}{13122\pi} g_4 \right. \\ &+ \left. (0.00943497 n^2 + 0.60941312 n + 7.15615323) g_4^2 \right]. \quad (7) \end{aligned}$$

we obtain:

$$\begin{aligned}
g_6^* = & \frac{8(n+26)\pi^2\tau^3}{3(n+8)^3} + \frac{\tau^4}{(n+8)^5}(181.308289 n^2 + 8340.18127 n + 26061.6043) \\
& + \frac{\tau^5}{(n+8)^7} \left( -78.4778860 n^4 - 1875.40831 n^3 + 37813.2081 n^2 + 191806.512 n \right. \\
& + 257751.564 \left. \right) - \frac{\tau^6}{(n+8)^9} \left( 10.616530 n^6 + 1095.9774 n^5 + 25502.145 n^4 \right. \\
& - 179690.51 n^3 - 616717.23 n^2 + 2241880.8 n + 7427442.9 \left. \right). \tag{8}
\end{aligned}$$

$$\begin{aligned}
g_8^* = & -\frac{8(n+80)\pi^3\tau^4}{(n+8)^4} + \frac{\tau^5}{(n+8)^6} \left( 248.050213 n^3 + 17743.2461 n^2 \right. \\
& + 77514.1600 n + 1072066.90 \left. \right) + \frac{\tau^6}{(n+8)^8} \left( 1387.95229 n^4 + 197852.837 n^3 \right. \\
& + 1715306.54 n^2 + 15922970.4 n + 8711448.94 \left. \right). \tag{9}
\end{aligned}$$

Since small magnetization expansion of free energy contains ratios of renormalized coupling constants

$$R_6 = \frac{g_6}{g_4^2}, \quad R_8 = \frac{g_8}{g_4^3} \tag{10}$$

rather than coupling constants themselves, it is reasonable to calculate pseudo- $\epsilon$  expansions for these ratios as well. They are as follows

$$\begin{aligned}
R_6^* = & \frac{2(n+26)\tau}{3(n+8)} - \frac{\tau^2}{(n+8)^3}(3.506172836 n^2 + 36.83950607 n + 315.6543210) \\
& + \frac{\tau^3}{(n+8)^5} \left( -0.18927773 n^4 + 6.51351435 n^3 + 396.321683 n^2 + 2777.67913 n \right. \\
& + 10998.4537 \left. \right) - \frac{\tau^4}{(n+8)^7} \left( 0.06139199 n^6 + 8.4167873 n^5 + 227.14320 n^4 \right. \\
& + 3434.7520 n^3 + 49684.392 n^2 + 283809.46 n + 691313.24 \left. \right). \tag{11}
\end{aligned}$$

$$\begin{aligned}
R_8^* = & -\frac{(n+80)\tau}{(n+8)} + \frac{\tau^2}{(n+8)^3}(n^3 + 89.75308641 n^2 + 1854.716049 n + 11077.53086) \\
& - \frac{\tau^3}{(n+8)^5} \left( 16.6736016 n^4 + 1111.20557 n^3 + 22512.7084 n^2 \right. \\
& + 199142.427 n + 713156.705 \left. \right). \tag{12}
\end{aligned}$$

### III. WILSON FIXED POINT LOCATION FROM THE PSEUDO- $\epsilon$ EXPANSION

Let us find numerical estimates for the fixed point value of quartic coupling constant at various  $n$  resulting from the pseudo- $\epsilon$  expansion (5). Address first the cases  $n = 0$ ,  $n = 1$ ,  $n = 2$  and  $n = 3$  that are known to correspond to physically realizable systems. Pseudo- $\epsilon$  expansions for the critical value of  $g_4$  we'll deal with are as follows:

$$g_4^* = \frac{\pi}{4} \left( \tau + 0.4398148148\tau^2 - 0.003048547\tau^3 + 0.015229668\tau^4 - 0.05740387\tau^5 + 0.0622931\tau^6 \right), \quad n = 0. \quad (13)$$

$$g_4^* = \frac{2\pi}{9} \left( \tau + 0.4224965707\tau^2 + 0.005937107\tau^3 + 0.011983594\tau^4 - 0.04123101\tau^5 + 0.0401346\tau^6 \right), \quad n = 1. \quad (14)$$

$$g_4^* = \frac{\pi}{5} \left( \tau + 0.4029629630\tau^2 + 0.009841357\tau^3 + 0.010593080\tau^4 - 0.02962102\tau^5 + 0.0282146\tau^6 \right), \quad n = 2. \quad (15)$$

$$g_4^* = \frac{2\pi}{11} \left( \tau + 0.3832262014\tau^2 + 0.010777962\tau^3 + 0.009577837\tau^4 - 0.02146532\tau^5 + 0.0211675\tau^6 \right), \quad n = 3. \quad (16)$$

The expansions for universal value of  $g_4$  were, in fact, analyzed earlier employing Borel transformation based resummation procedures<sup>3,19</sup>, although series (5), (13), (14), (15), (16) themselves, to our knowledge, have never been published. Here we resum these expansions and their counterparts for other  $n$  by means of Padé approximants [L/M], i. e. using rather simple approach. This technique is quite suitable in our situation since, as seen from (13)-(16), the pseudo- $\epsilon$  expansions have small higher-order coefficients. Padé approximant technique is widely known today (see, e. g. Ref.<sup>49</sup>), so we write down Padé tables for all four cases without going into detail. Two points, however, have to be mentioned. First, to make

comparison of our estimates with others more convenient, we present numerical results for rescaled constant  $g = g_4(n+8)/2\pi$ ; the series staying in brackets in (5), (13), (14), (15), (16) are precisely the pseudo- $\epsilon$  expansions for this constant. Second, Padé approximants are constructed for  $g^*/\tau$ , with factor  $\tau$  having physical value  $\tau = 1$  ignored. Tables I, II, III and IV present Padé triangles discussed.

It looks natural to adopt as a final estimate for  $g^*$  the average over two highest-order near diagonal Padé approximants  $[3/2]$  and  $[2/3]$ . We do so for all the cases of interest apart from  $n = 0$  when one of working approximants –  $[3/2]$  – has abnormally large higher-order coefficients (18.1, 41.2, 46.5) preventing obtaining high-precision estimate; corresponding number is marked in Table I with  $^+$ . In this case we accept as a most reliable the value given by another Padé approximant –  $[2/3]$ . So, our pseudo- $\epsilon$  expansion estimates for  $g^*$  are:

$$g^* = 1.423 \quad (n = 0), \quad g^* = 1.423 \quad (n = 1), \quad g^* = 1.410 \quad (n = 2), \quad g^* = 1.393 \quad (n = 3), \quad (17)$$

Numbers (17) differ from their canonical six-loop RG counterparts only in third or even in fourth ( $n = 3$ ) decimal place. This looks rather optimistic encouraging to work further with Padé resummed pseudo- $\epsilon$  expansions. Moreover, the accuracy of numerical results given by these series rapidly improves when dimensionality of the order parameter  $n$  grows up. To demonstrate this we present Padé triangle for  $n = 6$  (Table V). As seen from Table V, for this (not so big) value of  $n$  the numbers given by approximants  $[4/1]$ ,  $[3/2]$ , and  $[2/3]$  practically coincide with each other and with 6-loop RG estimate  $g^* = 1.3385^{24}$ .

Note that, as seen from Tables I–V, the numbers given by lower-order diagonal and near diagonal Padé approximants  $[2/2]$ ,  $[2/1]$ ,  $[1/2]$  are also close to asymptotic values of  $g$ . It means that the pseudo- $\epsilon$  expansion approach generates not only numerically efficient but rapidly converging iteration procedure.

The overall situation is illustrated by Table VI accumulating pseudo- $\epsilon$  expansion estimates of  $g^*$  for  $0 \leq n \leq 64$ . Along with Padé estimates (second column) the numbers obtained by direct summation of pseudo- $\epsilon$  expansions are presented here (third column). Direct summation is performed under the assumption that one can get best numerical estimates truncating divergent pseudo- $\epsilon$  expansions by smallest terms, i. e. adopting the procedure valid for asymptotic series. Padé estimates presented are the averages over those given by



near symmetric approximants  $[2/3]$  and  $[3/2]$ , apart from the case  $n = 0$  (see above) when the value given by another approximant is accepted as a final estimate. Numerical values of  $g^*$  resulting from analysis of 6-loop RG series in 3 dimensions<sup>19,24,35</sup> (fourth and fifth columns), obtained within the  $\epsilon$  expansion approach<sup>26,35</sup> (sixth column) and extracted from lattice calculations<sup>20,21</sup> (LC) are also collected in Table VI for comparison.

Table VI clearly demonstrates that the values of  $g^*$  obtained from Padé approximants and given by direct summation are very close to each other and to alternative high-precision estimates. Even for  $n = 1$  the difference between numbers produced by pseudo- $\epsilon$  expansion and by other advanced techniques, both field-theoretical and lattice, is of order of 0.01. This may be considered as a strong argument in favor of high numerical effectiveness of the pseudo- $\epsilon$  expansion approach. Moreover, direct summation of series for  $g^*$  generates an iteration procedure which, being quite primitive, rapidly converges to asymptotic values that are very close to most accurate estimates known today. In this sense, the pseudo- $\epsilon$  expansion approach itself may be referred to as some special resummation technique. To confirm or to disprove this statement, the structure of pseudo- $\epsilon$  expansions for other universal quantities and corresponding numerical estimates are to be analyzed.

#### IV. SEXTIC COUPLING AND UNIVERSAL RATIO $R_6$ FOR VARIOUS $n$

Let us estimate further universal critical values of  $g_6$  and  $R_6$  within the pseudo- $\epsilon$  expansion approach. For Ising, XY and Heisenberg models corresponding  $\tau$ -series read:

$n = 1$ :

$$g_6^* = \frac{8\pi^2}{81}\tau^3 + 0.585667731\tau^4 + 0.101488719\tau^5 - 0.0229712\tau^6. \quad (18)$$

$$R_6^* = 2\tau - 0.488340192\tau^2 + 0.240118863\tau^3 - 0.2150291\tau^4. \quad (19)$$

$n = 2$ :

$$g_6^* = \frac{28\pi^2}{375}\tau^3 + 0.434672000\tau^4 + 0.077635850\tau^5 - 0.0084506\tau^6. \quad (20)$$

$$R_6^* = \frac{28}{15}\tau - 0.403358025\tau^2 + 0.181881784\tau^3 - 0.1489055\tau^4. \quad (21)$$

$n = 3$ :

$$g_6^* = \frac{232\pi^2}{3993}\tau^3 + 0.327311986\tau^4 + 0.057293963\tau^5 - 0.0025831\tau^6. \quad (22)$$

$$R_6^* = \frac{58}{33}\tau - 0.343898118\tau^2 + 0.143177750\tau^3 - 0.1079237\tau^4. \quad (23)$$

Expansions for  $g_6^*$  are seen to have fast diminishing coefficients with irregular signs. On the contrary, coefficients of  $\tau$ -series for  $R_6^*$  decrease more slowly but these series are alternating. We'll concentrate on the numerical values of  $R_6^*$  which enters the scaling equation of state and has been estimated for various  $n$  within several field-theoretical and lattice methods<sup>16,22,25,26,35,38</sup>. Since higher-order coefficients of series (18) – (23) are rather small we do not need in Borel transformation killing factorial growth of coefficients and can process our series by means of Padé approximants or even perform their direct summation. To clear up to what extent numerical results are sensitive to the summation procedure we find the values of  $R_6^*$  in four different ways. Namely, we estimate  $R_6^*$

- i) by means of Padé summation of series for  $R_6^*$ ,
- ii) via Padé summation of series for  $g_6^*$  and use of the first relation (10),
- iii) by direct summation of  $R_6^*$  pseudo- $\epsilon$  expansion with optimal truncation, and
- iv) by optimally truncated direct summation (OTDS) of  $\tau$ -series for  $g_6^*$  and subsequent use of (10) with  $g_4^*$  also found by OTDS.

As was expected, Padé resummation turns out to be effective in our problem. One can see this from Padé triangles for  $g_6^*$  and  $R_6^*$  at  $n = 1$  presented in Tables VII and VIII. We choose here the Ising limit as an illustration not only because of its physical significance. More important point is that under  $n = 1$   $\tau$ -expansions for  $g_6^*$  and  $R_6^*$  have larger higher-order coefficients than those for  $n > 1$  making Ising model rather "unfriendly" for pseudo- $\epsilon$  expansion analysis.

Numerical results obtained for  $0 \leq n \leq 64$  are presented in Table IX. The second column contains universal values of  $R_6$  given by Padé summation of corresponding  $\tau$ -series. In the third column the estimates found via Padé summation of  $\tau$ -series for  $g_6^*$  are collected. The numbers obtained by optimally truncated direct summation of the series for  $R_6^*$  and  $g_6^*$  form fourth and fifth columns. Padé estimates reported in Table IX are those averaged over two near diagonal approximants  $[2/1]$  and  $[1/2]$  for  $R_6^*/\tau$  and  $g_6^*/\tau^3$ . When one of them is spoiled by a pole close to 1 or has abnormally large higher-order coefficients the value

given by another approximant is accepted as a final estimate; these numbers are marked with asterisks. The values of  $R_6^*$  resulting from 3D RG series<sup>16,25,26,35</sup>, obtained within  $\epsilon$ -expansion<sup>26,35</sup> and  $1/n$ -expansion<sup>22,50</sup> approaches and extracted from lattice calculations (LC) are also presented in the Table.

The values of  $R_6^*$  staying in second and fifth columns of Table IX are seen to be very close to each other and to RG estimates for any  $n$ . This fact may be understood keeping in mind the structure of pseudo- $\epsilon$  expansions for  $R_6^*$  and  $g_6^*$ . The series for  $R_6^*$  have small enough and monotonically decreasing coefficients with alternating signs what makes their summation by means of Padé approximants efficient<sup>49</sup>. The coefficients of the series for  $g_6^*$ , on the contrary, have irregular signs but their modulo decrease extremely rapidly and the last coefficients are tiny (see, e. g. (18), (20), (22)). This obviously favors direct summation. On the other hand,  $\tau$ -series for  $g_6^*$  because of fast decreasing coefficients are also suitable for Padé summation. That is why the numbers in the third column of Table IX are rather close to their counterparts from the second and fifth columns. In such a situation optimally truncated direct summation of  $\tau$ -series for  $R_6^*$  having no advantages looks as a crude procedure, at least when compared with others just discussed. Nevertheless, it provides quite satisfactory results for  $n \geq 10$  and leads to fair estimates for physical values of  $n$ .

So, we see that the pseudo- $\epsilon$  expansion approach combined with Padé resummation technique is a powerful instrument for analysis of effective sextic interaction at criticality. Moreover, even direct summation, if properly performed, is able to provide high-precision numerical estimates for the universal ratio  $R_6^*$  at any  $n$ .

## V. OCTIC COUPLING: STRUCTURE OF $\tau$ -SERIES AND NUMERICAL ESTIMATES

In the case of renormalized octic coupling we have shorter pseudo- $\epsilon$  expansions with much less favorable structure. This is clearly seen from the series for  $n = 1, 2, 3$  written below:

$n = 1$ :

$$g_8^* = -\frac{8\pi^3}{81}\tau^4 + 2.19699337\tau^5 + 0.616747712\tau^6 \quad (24)$$

$$R_8^* = -9\tau + 17.8641975\tau^2 - 15.8502213\tau^3. \quad (25)$$

n = 2:

$$g_8^* = -\frac{41\pi^3}{625}\tau^4 + 1.300052605\tau^5 + 0.490236460\tau^6. \quad (26)$$

$$R_8^* = -\frac{41}{5}\tau + 15.1539753\tau^2 - 12.1064882\tau^3. \quad (27)$$

n = 3:

$$g_8^* = -\frac{664\pi^3}{14641}\tau^4 + 0.830338865\tau^5 + 0.360948746\tau^6. \quad (28)$$

$$R_8^* = -\frac{83}{11}\tau + 13.1303207\tau^2 - 9.59044946\tau^3. \quad (29)$$

The series for  $R_8^*$  being alternating have big elder coefficients. That is why to estimate this ratio we apply, along with Padé resummation, Padé-Borel procedure. Higher-order coefficients of the expansions for  $g_8^*$  are much smaller. These series are processed within Padé technique on the base of approximant  $[1/1]$ , the only nontrivial and diagonal one existing for  $g_8^*/\tau^3$ . Then the value of  $R_8^*$  is estimated using the second relation (10).

Numerical results thus obtained are presented in Table X, along with the estimates of  $R_8^*$  deduced from RG series in three dimensions<sup>16,25,35</sup>, found within the  $\epsilon$ -expansion<sup>26,35</sup> and  $1/n$ -expansion<sup>22</sup> approaches and extracted from lattice calculations. As is seen, in the case of octic coupling numerical estimates turn out to be much worse than those obtained for  $g_4^*$  and  $R_6^*$ . Indeed, the numbers given by pseudo- $\epsilon$  expansions resummed in three different ways are strongly scattered, to say nothing about their marked deviation from estimates yielded by alternative methods. This is true not only for  $n = 0, 1, 2, 3$ , but even for  $n$  as large as 64: the difference between various pseudo- $\epsilon$  expansion estimates exceeds here 20%.

Of course, pronounced shortness and strong divergence of  $\tau$ -series for  $g_8^*$  and  $R_8^*$  may be thought of as main sources of such a failure. There exists, however, an extra moment making the situation quite unfavorable. The point is that the series (9), (12) have unusual feature. Namely, when  $n \rightarrow \infty$  the first and second terms in these expansions compensate each another diminishing their mutual contribution and increasing the role of higher-order terms; analogous peculiarity was observed earlier for original RG expansion of  $g_8^{25}$ . Since each of  $\tau$ -series (9), (12) is short and possesses only one such key higher-order term numerical effectiveness of pseudo- $\epsilon$  expansion turns out to be poor in this case. We believe that

calculation of the next terms in  $\tau$ -series for renormalized octic coupling would considerably improve the situation, as it occurs in other unfavorable cases<sup>41</sup>. To get longer  $\tau$ -series one needs, however, longer RG expansion for  $g_8$ . Today such an expansion is known only for the Ising model<sup>16</sup>.

## VI. RENORMALIZED QUARTIC COUPLING CONSTANT IN TWO DIMENSIONS

Here we'll apply the pseudo- $\epsilon$  expansion technique to estimate the critical values of quartic coupling constant for two-dimensional systems. Along with physically interesting cases  $n = 1$  and  $n = 0$  studied earlier<sup>36,44</sup> the models with  $n \geq 2$  will be considered. Although these models are known not to undergo phase transitions into ordered state, Wilson fixed point location has been calculated for them within field-theoretical 2D RG approach in five-loop approximation<sup>41</sup> and using  $\epsilon$ -expansions constrained at  $D = 0$  and  $D = 1$ <sup>21,26</sup>. Comparison of numerical results obtained within these techniques with those given by pseudo- $\epsilon$  expansions is believed to shed light on computational power of the latter approach.

Pseudo- $\epsilon$  expansion for fixed point value of  $g$  in 2D for arbitrary  $n$  is as follows<sup>44</sup>:

$$\begin{aligned}
g^* = & \tau + \frac{\tau^2}{(n+8)^2} \left( 10.33501055 n + 47.67505273 \right) \\
& + \frac{\tau^3}{(n+8)^4} \left( -5.00027593 n^3 + 24.4708201 n^2 + 253.297221 n + 350.808487 \right) \\
& + \frac{\tau^4}{(n+8)^6} \left( 0.088842906 n^5 - 77.270445 n^4 + 45.052398 n^3 + 3408.2839 n^2 \right. \\
& + 14721.151 n + 27649.346 \left. \right) - \frac{\tau^5}{(n+8)^8} \left( -0.00407946 n^7 - 0.305739 n^6 \right. \\
& + 1464.58 n^5 + 11521.4 n^4 + 98803.3 n^3 + 794945 n^2 + 3146620 n + 4734120 \left. \right). \quad (30)
\end{aligned}$$

We estimate  $g^*$  for various  $n$  lying between 0 and 64 within Padé and Padé-Borel resummation techniques and by optimally truncated direct summation. In course of Padé resummation the approximant  $[3/2]$  is used apart from the cases when it has poles close to 1. The choice of approximant  $[3/2]$  is quite natural since it is equivalent to diagonal approximant  $[2/2]$  for  $g^*/\tau$ , i. e. with insignificant factor  $\tau$  neglected. Padé-Borel resummation is based on highest-order approximants having no positive axis ("dangerous") poles that pre-

vent evaluation of Borel integral. Direct summation is performed as before with truncation on the term with smallest coefficient.

The results obtained are collected in Table XI. The fixed point values of  $g$  resulting from 5-loop RG series<sup>41</sup>, obtained within constrained  $\epsilon$ -expansion approach<sup>26</sup> and extracted from  $(1/n)$ -expansion<sup>11</sup> and lattice calculations<sup>13,28</sup> (LC) are also presented there to compare with our data. As is seen from Table XI pseudo- $\epsilon$  expansion results in quite good numerical estimates for any  $n$  provided Padé or Padé-Borel resummation is made. In fact, use of Padé-Borel resummation changes numerical estimates only slightly leaving simple Padé procedure efficient in two dimensions. Moreover, even direct summation remains satisfactory at the quantitative level down to  $n = 4$  leading as well to reasonable numbers in physical cases  $n = 1$  and  $n = 0$ . So, estimating renormalized quartic coupling constant in two dimensions on the base of pseudo- $\epsilon$  expansion one can use simplest ways to process the series - Padé approximants and direct summation.

These results lead us, as above, to the conclusion that the pseudo- $\epsilon$  expansion itself may be considered as a resummation method. The first argument in favor of such a point of view is obvious: this approach turns strongly divergent field-theoretical RG expansions into power series with smaller lower-order coefficients and much slower increasing higher-order ones. The second argument is specific for low-dimensional systems: the physical value of the pseudo- $\epsilon$  expansion parameter  $\tau$  is equal to 1, while the Wilson fixed point coordinate  $g^*$  playing analogous role within field-theoretical RG approach is almost two times bigger for physical values of  $n$  ( $g^* \approx 1.8$ ). This difference is essential, especially keeping in mind importance of higher-order terms<sup>41</sup>.

## VII. CONCLUSION

To summarize, we have calculated pseudo- $\epsilon$  expansions for universal values of renormalized coupling constants  $g_4$ ,  $g_6$ ,  $g_8$  and of ratios  $R_6$ ,  $R_8$  for 3D Euclidean  $n$ -vector  $\lambda\phi^4$  field theory. Numerical estimates for Wilson fixed point location  $g_4^*$  and for  $R_6^*$  and  $R_8^*$  have been found under  $0 \leq n \leq 64$  using Padé and Padé-Borel resummation techniques as well as by direct summation with optimal truncation. For  $g_4^*$  and  $R_6^*$  pseudo- $\epsilon$  expansion machinery was shown to lead to high-precision numerical estimates without addressing Borel transformation. Moreover, in both cases properly performed direct summation turned out to be

sufficient to result in accurate enough numbers at any  $n$ . This implies that the pseudo- $\epsilon$  expansion approach itself may be thought of as some specific resummation technique. For the octic coupling, however, this technique was shown to be much less efficient: numerical estimates found by Padé and Padé-Borel summation of  $\tau$ -series for  $R_8^*$  and obtained via evaluation of  $g_8^*$  are strongly scattered and considerably deviate from their lattice and field-theoretical counterparts. This failure, however, does not indicate poor numerical effectiveness of the pseudo- $\epsilon$  expansion approach; it is caused mainly by shortness of corresponding  $\tau$ -series and the unfavorable feature of their structure.

Pseudo- $\epsilon$  expansion for renormalized quartic coupling constant of 2D  $n$ -vector field theory has been also analyzed. Universal values of  $g_4$  for  $0 \leq n \leq 64$  have been estimated using Padé and Padé-Borel resummation techniques as well as by direct summation with optimal cut off. Comparison of the results obtained with each other and with their counterparts known from alternative field-theoretical and lattice calculations has shown that pseudo- $\epsilon$  expansion technique provides numerical estimates as accurate as those given by other advanced approaches.

### Acknowledgment

We dedicate our work to the memory of Kenneth Wilson whose talk at the Soviet-American symposium in Leningrad in 1971 inspired one of us (A. I. S.) to enter the Realm of Renormalization Group.

---

\* Electronic address: ais2002@mail.ru

<sup>1</sup> G. A. Baker, Phys. Rev. B **15**, 1552 (1977).

<sup>2</sup> G. A. Baker, B. G. Nickel, and D. I. Meiron, Phys. Rev. B **17**, 1365 (1978).

<sup>3</sup> J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980).

<sup>4</sup> C. Bagnuls and C. Bervillier, Phys. Rev. B **41**, 402 (1990).

<sup>5</sup> C. M. Bender, S. Boettcher, and L. Lipatov, Phys. Rev. Lett. **68**, 3674 (1992).

<sup>6</sup> C. M. Bender and S. Boettcher, Phys. Rev. D **48**, 4919 (1993).

<sup>7</sup> N. Tetradis and C. Wetterich, Nucl. Phys. B **422**, 541 (1994).

<sup>8</sup> C. M. Bender and S. Boettcher, Phys. Rev. D **51**, 1875 (1995).

- <sup>9</sup> T. Reisz, Phys. Lett. B **360**, 77 (1995).
- <sup>10</sup> S. A. Antonenko and A. I. Sokolov, Phys. Rev. E **51**, 1894 (1995).
- <sup>11</sup> M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Nucl. Phys. B **459**, 207 (1996).
- <sup>12</sup> A. I. Sokolov, Fiz. Tverd. Tela **38**, 640 (1996) [Phys. Solid State **38**, 354 (1996)].
- <sup>13</sup> P. Butera and M. Comi, Phys. Rev. B **54**, 15828 (1996).
- <sup>14</sup> S.-Y. Zinn, S.-N. Lai, and M. E. Fisher, Phys. Rev. E **54**, 1176 (1996).
- <sup>15</sup> A. I. Sokolov, E. V. Orlov, V. A. Ul'kov, Phys. Lett. A **227**, 255 (1997).
- <sup>16</sup> R. Guida and J. Zinn-Justin, Nucl. Phys. B **489**, 626 (1997).
- <sup>17</sup> T. R. Morris, Nucl. Phys. B **495**, 477 (1997).
- <sup>18</sup> P. Butera and M. Comi, Phys. Rev. E **55**, 6391 (1997).
- <sup>19</sup> R. Guida and J. Zinn-Justin, J. Phys. A **31**, 8103 (1998).
- <sup>20</sup> P. Butera and M. Comi, Phys. Rev. B **58**, 11552 (1998).
- <sup>21</sup> A. Pelissetto and E. Vicari, Nucl. Phys. B **519**, 626 (1998).
- <sup>22</sup> A. Pelissetto and E. Vicari, Nucl. Phys. B **522**, 605 (1998).
- <sup>23</sup> A. I. Sokolov and E. V. Orlov, Phys. Rev. B **58**, 2395 (1998).
- <sup>24</sup> A. I. Sokolov, Fiz. Tverd. Tela **40**, 1284 (1998) [Phys. Solid State **40**, 1169 (1998)].
- <sup>25</sup> A. I. Sokolov, E. V. Orlov, V. A. Ul'kov, and S. S. Kashtanov, Phys. Rev. E **60**, 1344 (1999).
- <sup>26</sup> A. Pelissetto and E. Vicari, Nucl. Phys. B **575**, 579 (2000).
- <sup>27</sup> J. Zinn-Justin, Phys. Reports **344**, 159 (2001).
- <sup>28</sup> M. Campostrini, M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Phys. A **34**, 2923 (2001).
- <sup>29</sup> M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B **63**, 214503 (2001).
- <sup>30</sup> M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B **65**, 144520 (2002).
- <sup>31</sup> M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. E **65**, 066127 (2002).
- <sup>32</sup> J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Clarendon Press, Oxford, 2002).
- <sup>33</sup> A. Pelissetto and E. Vicari, Phys. Reports **368**, 549 (2002).
- <sup>34</sup> F. P. Toldin, A. Pelissetto, and E. Vicari, J. High Energy Phys. **07**, 029 (2003).
- <sup>35</sup> A. Butti, F. P. Toldin, Nucl. Phys. B **704**, 527 (2005).
- <sup>36</sup> A. I. Sokolov, Fiz. Tverd. Tela **47**, 2056 (2005) [Phys. Sol. State **47**, 2144 (2005)]



- <sup>37</sup> A. Pelissetto and E. Vicari, J. Phys. A **40**, F539 (2007).
- <sup>38</sup> P. Butera and M. Pernici, Phys. Rev. B **83**, 054433 (2011).
- <sup>39</sup> D. V. Pakhnin and A. I. Sokolov, Phys. Rev. B **61**, 15130 (2000).
- <sup>40</sup> J. M. Carmona, A. Pelissetto, and E. Vicari, Phys. Rev. B **61**, 15136 (2000).
- <sup>41</sup> E. V. Orlov and A. I. Sokolov, Fiz. Tverd. Tela **42**, 2087 (2000) [Phys. Sol. State **42**, 2151 (2000)]
- <sup>42</sup> P. Calabrese, E. V. Orlov, D. V. Pakhnin, and A. I. Sokolov, Phys. Rev. B **70**, 094425 (2004).
- <sup>43</sup> I. O. Mayer, A. I. Sokolov and B. N. Shalaye, Ferroelectrics **95**, 93 (1989).
- <sup>44</sup> M. A. Nikitina and A. I. Sokolov, arXiv:1312.1062.
- <sup>45</sup> R. Folk, Yu. Holovatch, and T. Yavorskii, Phys. Rev. B **62**, 12195 (2000).
- <sup>46</sup> Yu. Holovatch, D. Ivaneiko, and B. Delamotte, J. Phys. A **37**, 3569 (2004).
- <sup>47</sup> P. Calabrese and P. Parruccini, Phys. Rev. B **71**, 064416 (2005).
- <sup>48</sup> B. G. Nickel, D. I. Meiron, and G. A. Baker, University of Guelph Report, 1977.
- <sup>49</sup> G. A. Baker, Jr. and P. Graves-Morris. Padé Approximants (Addison-Wesley, Reading, MA, 1981).
- <sup>50</sup> E. Brezin and D. J. Wallace, Phys. Rev. B **7**, 1967 (1973).

TABLE I: Padé table for pseudo- $\epsilon$  expansion of quartic coupling constant  $g^* = g_4^*(n+8)/2\pi$  at  $n = 0$  (self-avoiding walks). Padé approximants  $[L/M]$  are derived for  $g^*/\tau$ , i. e. with factor  $\tau$  omitted. Approximant  $[3/2]$  has abnormally large higher-order coefficients (18.1, 41.2 and 46.5) preventing obtaining high-precision estimate; corresponding number is marked with  $^+$ . The value of  $g^*$  resulting from resummed original six-loop RG series and referred to as most reliable RG estimate is equal to  $1.413 \pm 0.006^{19}$ .

$M \setminus L$	0	1	2	3	4	5
0	1	1.4398	1.4368	1.4520	1.3946	1.4569
1	1.7851	1.4368	1.4393	1.4400	1.4245	
2	1.3216	1.4512	1.4400	1.4394 $^+$		
3	1.5298	1.4147	1.4226			
4	1.3094	1.4240				
5	1.6014					

TABLE II: Padé triangle for pseudo- $\epsilon$  expansion of coupling constant  $g^* = g_4^*(n+8)/2\pi$  at  $n = 1$  (Ising model). Padé approximants  $[L/M]$  are derived for  $g^*/\tau$ , i. e. with factor  $\tau$  omitted. The value of  $g^*$  given by resummed six-loop RG series is equal to  $1.411 \pm 0.004^{19}$ .

$M \setminus L$	0	1	2	3	4	5
0	1	1.4225	1.4284	1.4404	1.3992	1.4393
1	1.7316	1.4285	1.4167	1.4311	1.4195	
2	1.3332	1.4403	1.4313	1.4273		
3	1.4977	1.4118	1.4179			
4	1.3373	1.4190				
5	1.5272					

TABLE III: The same as Table II but for  $n = 2$  (XY model). Approximant  $[2/1]$  has a pole close to 1; its location is shown as a subscript. Six-loop RG estimate for  $g^*$  is  $1.403 \pm 0.003^{19}$ .

$M \setminus L$	0	1	2	3	4	5
0	1	1.4030	1.4128	1.4234	1.3938	1.4220
1	1.6749	1.4131	1.2741 <sub>0.93</sub>	1.4156	1.4082	
2	1.3341	1.4235	1.4159	1.4119		
3	1.4674	1.4019	1.4072			
4	1.3490	1.4082				
5	1.4791					

TABLE IV: The same as Table II but for  $n = 3$  (Heisenberg model). Approximant  $[2/1]$  has a pole close to 1; its location is shown as a subscript. Six-loop RG estimate for  $g^*$  is  $1.390 \pm 0.004^{19}$ .

$M \setminus L$	0	1	2	3	4	5
0	1	1.3832	1.3940	1.4036	1.3821	1.4033
1	1.6213	1.3943	1.4800 <sub>1.13</sub>	1.3970	1.3928	
2	1.3283	1.4037	1.3973	1.3943		
3	1.4383	1.3874	1.3922			
4	1.3495	1.3934				
5	1.4422					

TABLE V: The same as Table II but for  $n = 6$ . Approximant  $[2/1]$  has a pole very close to 1; its location is shown as a subscript. The value of  $g^*$  given by Padé-Borel resummed six-loop RG series is  $1.3385^{24}$ .

$M \setminus L$	0	1	2	3	4	5
0	1	1.3296	1.3362	1.3426	1.3335	1.3444
1	1.4915	1.3363	1.5821 <sub>1.03</sub>	1.3389	1.3385	
2	1.2946	1.3426	1.3390	1.3385		
3	1.3614	1.3353	1.33845			
4	1.3200	1.3398				
5	1.3593					

TABLE VI: Fixed point values of quartic coupling constant  $g$  for various  $n$  found by Padé summation of corresponding pseudo- $\epsilon$  expansions and by direct summation of these series with the optimal cut off (OTDS). Padé estimates are averages over those given by near diagonal approximants  $[2/3]$  and  $[3/2]$ . Since for  $n = 0$  approximant  $[3/2]$  has abnormally large higher-order coefficients (see caption to Table I) the value given by another approximant is accepted as a final estimate; it is marked by asterisk. The universal values of  $g$  resulting from 6-loop RG series in 3 dimensions<sup>19,24,35</sup>, obtained within the  $\epsilon$ -expansion approach<sup>26,35</sup> and extracted from lattice calculations<sup>20,21</sup> (LC) are presented for comparison.

$n$	Padé	OTDS	3D RG <sup>24</sup>	3D RG	$\epsilon$ -exp. <sup>26</sup>	LC <sup>20</sup>	LC <sup>21</sup>
0	1.423*	1.437		1.413(6) <sup>19</sup>	1.396(20)	1.388(5)	1.393(20)
1	1.423	1.428	1.419	1.411(4) <sup>19</sup>	1.408(13)	1.408(7)	1.406(9)
2	1.410	1.413	1.4075	1.403(3) <sup>19</sup>	1.425(24)	1.411(8)	1.415(11)
3	1.393	1.404	1.392	1.390(4) <sup>19</sup>	1.426(9)	1.409(10)	1.411(12)
4	1.375	1.383	1.3745	1.377(5) <sup>19</sup>	1.393(21)	1.392(10)	1.396(16)
5	1.357	1.362	1.3565	1.3569 <sup>35</sup>	1.345 <sup>35</sup>		
6	1.3385	1.3426	1.3385	1.3397 <sup>35</sup>	1.321 <sup>35</sup>	1.355(10)	
8	1.3043	1.3024	1.3045		1.307(6)	1.320(15)	1.321(10)
10	1.2743	1.2733	1.2745			1.290(15)	
16	1.2075	1.2090	1.2077		1.202(4)		1.215(5)
24	1.1540	1.1542	1.1542		1.150(4)		1.158(4)
32	1.1215	1.1216	1.1218	1.1219 <sup>35</sup>	1.119(3)		1.122(3)
40	1.1001	1.1003	1.1003				
48	1.0850	1.0852			1.085(2)		1.084(2)
64	1.0652	1.0655		1.0656 <sup>35</sup>	1.0638 <sup>35</sup>		

TABLE VII: Padé triangle for pseudo- $\epsilon$  expansion of sextic coupling constant  $g_6^*$  for Ising model.

Padé approximants  $[L/M]$  are derived for  $g_6^*/\tau^3$ , i. e. with factor  $\tau^3$  omitted.

$M \setminus L$	0	1	2	3
0	0.9748	1.5604	1.6619	1.6390
1	2.4420	1.6832	1.6432	
2	1.4858	1.6188		
3	1.6582			

TABLE VIII: Padé triangle for pseudo- $\epsilon$  expansion of universal ratio  $R_6^*$  for Ising model. Padé

approximants  $[L/M]$  are derived for  $R_6^*/\tau$ , i. e. with factor  $\tau$  omitted.

$M \setminus L$	0	1	2	3
0	2	1.5117	1.7518	1.5367
1	1.6075	1.6726	1.6383	
2	1.6896	1.6465		
3	1.6036			

TABLE IX: Universal values of  $R_6$  for various  $n$  found by Padé summation of corresponding  $\tau$ -series (second column), obtained via Padé summation of  $\tau$ -series for  $g_6$  (third column), given by optimally truncated direct summation of the series for  $R_6^*$  (fourth column), and obtained via directly summed up  $\tau$ -series for  $g_6^*$  and  $g_4^*$  (fifth column). Padé estimates are those averaged over approximants  $[2/1]$  and  $[1/2]$  for  $R_6^*/\tau$  and  $g_6^*/\tau^3$ . If one of them suffers from some pathology (see text) the estimate given by another approximant is accepted as a final one; these numbers are marked with asterisks. The values of  $R_6^*$  resulting from 3D RG series<sup>16,25,26,35</sup>, obtained within  $\epsilon$ -expansion<sup>26,35</sup> and  $1/n$ -expansion<sup>22,50</sup> approaches and extracted from lattice calculations (LC) are presented for comparison.

$n$	Padé	Padé for $g_6$	OTDS	OTDS for $g_6$ and $g_4$	3D RG <sup>25</sup>	3D RG	$\epsilon$ -exp. <sup>26</sup>	LC and ( $1/n$ )-exp. ( $n \geq 8$ )
0	1.726	1.733	1.556	1.727		1.69(7) <sup>26</sup>	1.718(18)	
1	1.642	1.654	1.537	1.649	1.648	1.644(6) <sup>16</sup>	1.652(15)	1.649 <sup>38</sup>
2	1.566	1.576	1.496	1.574	1.574	1.576(10) <sup>26</sup>	1.575(10)	1.560(12) <sup>29</sup>
3	1.497	1.505	1.449	1.486	1.504	1.507(26) <sup>26</sup>	1.494(8)	1.49(3) <sup>30</sup>
4	1.436	1.439	1.401	1.427	1.442	1.447(22) <sup>26</sup>	1.424(7)	1.5(5) <sup>9</sup>
5	1.381	1.384	1.355	1.375	1.387	1.38(2) <sup>35</sup>	1.36(1) <sup>35</sup>	
6	1.3325	1.335	1.312	1.327	1.338	1.33(2) <sup>35</sup>	1.31(2) <sup>35</sup>	
8	1.2505	1.2511	1.237	1.2554	1.254		1.230(12)	1.688
10	1.1849	1.1851	1.1752	1.1873	1.187			1.484
16	1.0508	1.0506*	1.0454	1.0567	1.050		1.040(15)	1.177
24	0.9506	0.9508*	0.9464	0.9504	0.948			1.007
32	0.8899*	0.8883	0.8877	0.8912	0.889	0.8885(6) <sup>35</sup>	0.889(8)	0.922
40	0.8510*	0.8504	0.8525	0.8519	0.848			0.871
48	0.8236*	0.8234	0.8245	0.8243			0.823(4)	0.837
64	0.7877*	0.7876	0.7879	0.7879		0.7855(3) <sup>35</sup>	0.7877(3) <sup>35</sup>	0.794

TABLE X: The values of  $R_8^*$  for various  $n$  found by Padé summation of corresponding pseudo- $\epsilon$  expansions (second column), by Padé-Borel summation of these series (third column) and obtained via Padé summation of the series for  $g_8^*$  (fourth column). The values of  $R_8^*$  resulting from RG series in 3 dimensions<sup>16,25,35</sup>, obtained within the  $\epsilon$ -expansion<sup>26,35</sup> and  $1/n$ -expansion<sup>22</sup> approaches and extracted from lattice calculations (LC) are presented for comparison.

$n$	Padé	Padé-Borel	$R_8$ via $g_8$	3D RG <sup>25</sup>	$\epsilon$ -exp. <sup>26</sup>	LC	$(1/n)$ -exp.
0	0.786	1.614	-0.160		1.1(2)		
1	0.466	1.100	-0.008	0.856	0.94(14)	0.871(14) <sup>38</sup>	
				0.857(86) <sup>16</sup>	0.78(5) <sup>16</sup>	0.79(4) <sup>31</sup>	
2	0.224	0.726	0.076	0.563	0.71(16)	0.494(34) <sup>29</sup>	
3	0.043	0.450	0.124	0.334	0.33(10)	0.21(7) <sup>30</sup>	
4	-0.094	0.244	0.134	0.15	0.065(80)	0.07(14) <sup>34</sup>	
5	-0.196	0.089	0.113	-0.3(9) <sup>35</sup>	-0.1(2) <sup>35</sup>		
6	-0.273	-0.029	0.074	-0.09	-0.2(1) <sup>35</sup>		
8	-0.374	-0.189	-0.022	-0.25	-0.405(31)		-2.885
16	-0.475	-0.391	-0.254	-0.44	-0.528(14)		-1.442
32	-0.398	-0.365	-0.289	-0.42	-0.425(7)		-0.721
				-0.45(7) <sup>35</sup>	-0.427(3) <sup>35</sup>		
48	-0.319	-0.301	-0.247		-0.322(2)		-0.481
64	-0.263	-0.252	-0.209	-0.29(3) <sup>35</sup>	-0.269(3) <sup>35</sup>		-0.361

TABLE XI: The values of quartic coupling  $g^*$  in two dimensions for various  $n$  found by Padé and Padé-Borel resummation of corresponding pseudo- $\epsilon$  expansions and by direct summation of these series with optimal truncation (OTDS). Padé estimates are those given by approximant  $[3/2]$ , i. e. by diagonal approximant  $[2/2]$  for  $g^*/\tau$ . When this approximant has pole close to 1 approximant  $[2/3]$  (marked by subscript) is used. Padé-Borel estimates are based on approximants free of dangerous – positive axis – poles; relevant approximants are shown as subscripts. For  $n = 0$  and  $n = 1$  the numbers yielded by two different working approximants are presented to give an idea about the level of accuracy of the iteration scheme employed. The fixed point values of  $g$  resulting from 5-loop RG series in two dimensions<sup>41</sup>, obtained within the  $\epsilon$ -expansion<sup>26</sup> and  $(1/n)$ -expansion<sup>11</sup> approaches and extracted from lattice calculations<sup>13,28</sup> (LC) are presented for comparison.

$n$	Padé	Padé-Borel	OTDS	Constr. $\epsilon$ -exp. <sup>26</sup>	2D RG <sup>41</sup>	LC <sup>13</sup>	$(1/n)$ -exp. <sup>11</sup>
0	1.872	1.862 <sub>[4/1]</sub>	1.831	1.72(4)	1.86(4)	1.676(3)	
	1.749 <sub>[2/3]</sub>	1.710 <sub>[2/3]</sub>					
1	1.850	1.839 <sub>[4/1]</sub>	1.897	1.76(5)	1.84(3)	1.7538(5)	
	1.751 <sub>[2/3]</sub>	1.710 <sub>[2/3]</sub>				1.754365(3) <sup>28</sup>	
2	1.809	1.799 <sub>[4/1]</sub>	1.845	1.82(3)	1.80(3)	1.82(1)	
3	1.759	1.751 <sub>[4/1]</sub>	1.787	1.75(3)	1.75(2)		1.759
4	1.707	1.712 <sub>[3/2]</sub>	1.729	1.67(4)	1.70(2)	1.66(1)	1.699
8	1.531	1.532 <sub>[3/2]</sub>	1.535	1.46(3)	1.52(1)	1.43(3)	1.480
16	1.303 <sub>[2/3]</sub>	1.308 <sub>[2/3]</sub>	1.321	1.28(2)	1.313(3)		1.283
24	1.213 <sub>[2/3]</sub>	1.219 <sub>[2/3]</sub>	1.206	1.20(2)			1.200
32	1.163 <sub>[2/3]</sub>	1.168 <sub>[2/3]</sub>	1.158	1.16(1)	1.170(2)		1.154
48	1.105	1.114 <sub>[2/3]</sub>	1.107	1.11(1)			1.106
64	1.0806	1.0859 <sub>[2/3]</sub>	1.0815				1.0806